An Introduction to Gravitational Waves

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Abstract
This paper presents a brief overview of gravitational waves. Their propagation and generation are presented in more detail, with references to detailed derivations. The reader is assumed to be familiar with basic concepts of general relativity, but a brief review of relevant concepts is provided.

1 Introduction

From general relativity, gravity can be expressed as space-time curvature caused by the presence of mass [1]. Quadrupole accelerations of mass distributions will produce ripples in space-time. These ripples propagate at the speed of light, and are known as gravitational waves.

Gravitational waves were first discussed by Laplace in 1805, when he explored the results of hypothetical finite-speed gravitational influence [2]. He predicted that the angular momentum of a binary star system would decrease with time, a result that would now be interpreted as angular momentum being carried away by gravitational waves.

The modern form of gravitational waves was first presented in Einstein's 1915 publication of general relativity [1], as wave-like solutions to a linearized form of the equations. They were not widely studied until the 1950s, when it was proved by Hermann Bondi that gravitational waves are physically observable and in fact carry energy [2].

The only confirmed evidence for gravitational waves, so far, is the observed decay of a binary pulsar system matching the predictions of decay by emission of gravitational radiation. A Nobel Prize was awarded in 1993 for this discovery [2]. While there are several ongoing efforts, gravitational waves have not yet been directly detected.

Gravitational waves have the potential to be very useful in the field of astronomy. Being very different in nature from electromagnetic waves, they can reveal otherwise unobservable information. As they have minimal interaction with matter, they can penetrate barriers, such as dust clouds, that would scatter or absorb electromagnetic emissions. Being produced by coherent mass movements, they also reveal the internal structure of massive objects, whereas electromagnetic radiation usually carries incoherent surface information.

2 General Relativity

In general relativity, space-time is a four-dimensional surface where gravity is a result of curvature. Space-time curves in response to matter, and matter moves in response to the curvature. This curvature of space is given by the symmetric second-rank metric tensor *g*. The metric defines the notion of distance within a space, describing the vector magnitude:

$$|x| = x^{\mu} g_{\mu\nu} x^{\nu}$$

where x^{μ} is the 4-vector (t, x, y, z). The Minkowski metric $\eta_{\mu\nu}$ represents flat (0-curvature) space:

$$\eta = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The Einstein Field Equations (EFE) define the relationship between the metric tensor and the energy-momentum tensor:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T^{\mu\nu}$$

The energy-momentum tensor *T* is a measure of the presence of matter and energy in the local space. $R_{\mu\nu}$ is the Ricci tensor, and *R* is the scalar curvature. These can be qualitatively viewed as the Laplacian of the metric tensor *g* [4 appendix B].

3 Propagation

For the purposes of this discussion, gravitational waves will only be considered in an otherwise flat space-time. That is, they will be treated as weak perturbations of the flat Minkowski space described above. Using this perturbation analysis, gravitational waves are transverse quadrupole plane waves, with a velocity of c. The waves are area preserving in the transverse plane, such that if they expand space-time along one transverse direction, they will compress it in the other. This results in two orthogonal polarizations, known as the "plus" and "cross" polarizations, offset by 45 degrees (Figure 1).

Specifically, gravitational waves can be approximated as a small perturbation $h_{\mu\nu}$ of the flat-space Minkowski metric $\eta_{\mu\nu}$:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \text{ where } |h_{\mu\nu}| << 1$$

Expanding Einstein's field equations in h, they can be linearized (see appendix B of reference [4] for a derivation) such that in the Lorentz gauge $h_{\mu\nu}$ can be related to the stress-energy tensor:

$$\Box \bar{h}_{\mu\nu} = -16\pi G T^{\mu\nu} \tag{1}$$

with □ being the D'Alembertian.

In the limit of flat space-time as specified initially, where the perturbation is small and no significant mass is present, $T^{\mu\nu} \approx 0$, so

$$\Box \bar{h}_{\mu\nu} = 0 \tag{2}$$

This is recognizable as a wave equation. Solving for $h_{\mu\nu}$ as a traveling transverse wave in the +z direction at velocity c results in

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{ikx}$$

where k is a constant 4-vector. h_+ and $h_\times$ are the two possible polarizations of the gravitational wave. See reference [4] for a derivation of this solution.

As a gravitational wave passes a point, the perturbation of the local metric deforms space. This may cause the separation between nearby points to change. The effect of each polarization is shown in Figure 1. If a + polarized wave interacts with two particles on the x-axis as shown, the space between them will expand and contract at the frequency of the gravitational wave, and the separation of the two points along the y-axis will behave similarly, but out of phase by π. Points on the x and y axes would be unaffected by a × polarized wave.

4 Generation

The dominant producer of gravitational waves is a change of the quadrupole moment of the mass distribution. Because there is a single charge of mass, the dipole will always be the center of mass of the system, which is constant by conservation of momentum. Therefore, only asymmetric accelerations in a system of mass will produce gravitational waves.

Instead of solving for the massless equation, the linearized EFE can be written with nonzero T:

$$\bar{h}_{\mu\nu} = -16\pi G T^{\mu\nu}$$

This can be solved via a multipole expansion. Taking the lowest-order contribution,

$$h_{ij} \approx \frac{2G}{c^4 r}\left(\frac{d^2}{dt^2} Q_{ij}\right)_{retarded}$$

where the time derivatives are taken at the retarded time $t - r / c$, r is the distance to the source, Q_{ij} is the quadrupole moment, and h is the amplitude. See reference [4] for a detailed derivation of this.

5 Sources

Possible sources of gravitational waves, therefore, must by asymmetrically oscillating. Especially interesting candidates are compact binaries, such as a binary neutron star or black hole. Because of the compact size of these objects, the second time derivative of the quadrupole moment can be very large, and therefore will present the best chance of detection. At least one such binary has been optically observed to be loosing angular momentum, presumably due to gravitational radiation, and detection via gravitational waves would provide an additional test of general relativity.

Most moving astronomical objects with asymmetric mass distributions can emit gravitational waves. Other interesting astronomical objects with large second-derivative quadrupole terms are rotating neutron stars with asymmetric mass distribution, collapsing supernovae with nonuniform initial density, and merging black holes.

6 Detection

Detection of gravitational waves is extremely difficult. While sources emit enormous amount of energy, space-time is elastically stiff, and therefore disturbances are very small. For example, if a source had an off-axis kinetic energy of one solar mass and was located in the Virgo cluster, $h \le 10^{-21}$ by the equation presented in section 3 [5]. This corresponds to a proportional length change in a detector, $h/2 = \Delta L / L$, requiring extremely precise detectors.

Many efforts have been made to detect gravitational waves, but none has yet met with success. In the past, most experiments were resonant bar detectors. These consist of a large, precisely sized metal bar, at a known temperature. As gravitational waves pass, they could stretch the metal bar, whose lattice would resist the change. The resonant bar detectors were designed to sustain a resonance if this happened, enabling detection of the passing. The first resonant bar was constructed in the early 1970's by Joseph Weber, and had a strain sensitivity of approximately 10^{-15}, far too high for potential sources. No reproducible results have been found with this method.

Most recent detectors use interferometry. Michelson interferometers with very long arms attempt to detect length changes over these arms, which would occur during the passage of a gravitational wave. Early designs with 20-40m arms had strain sensitivities on the order of 10^{-20}, still too low to distinguish a signal. Current efforts have much larger arms, and projected strain sensitivities around 10^{-22}. LIGO, the Laser Interferometer Gravitational wave Observatory, is one such project. It has several interferometer sites, one in Hanford, WA with 2km arms. A similar project, GEO600, is based in Germany with 600m arms, and VIRGO in France has 3km arms. All these facilities are currently undergoing commissioning, and results are expected soon. The first space-based GW interferometer, LISA, is planned to launch in 2013, with arm lengths of 5 million km. It has a projected sensitivity of around 10^{-23}, and should definitely detect gravitational waves.

7 Conclusion

Gravitational waves can be treated as perturbations of the local space-time metric. Solving for these perturbations in otherwise flat space-time (no appreciable matter or energy present) using a linearized form of the Einstein Field Equations results in transverse traveling wave solutions with two polarizations, and velocity *c*. If the same linearized EFE are solved with a nonzero matter-energy tensor, the lowest-order term is due to quadrupole acceleration.

Figures

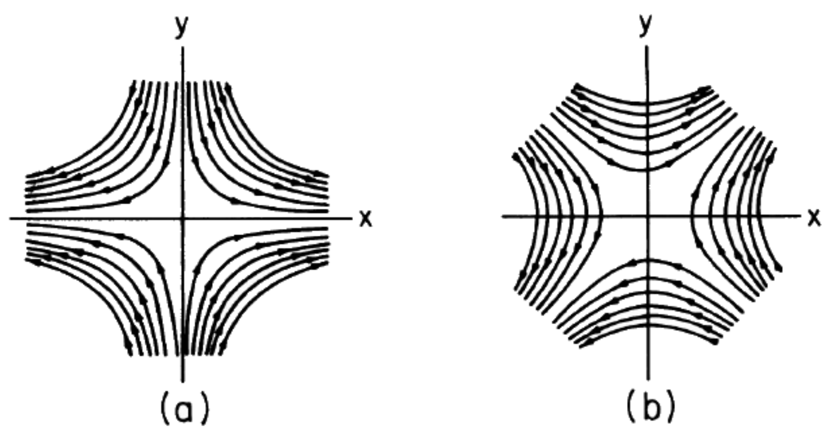

Figure 1: Plus (+) and cross (×) polarizations of gravitational waves

References

[1] A. Einstein, "On the general theory of relativity", Sitzungsberichte Preußische Akademie der Wissenschaften Berlin (Math. Phys.) 778 (1915).

[2] A. Hammesfahr et al., "LISA, Laser Interferometer Space Antenna: A Cornerstone Mission for the Observation of Gravitational Waves, System and Technology Study Report", as white book (2000) 7-16.

[3] J. B. Camp and N. J. Cornish, "The Status of Gravitational Wave Astronomy", for LIGO Scientific Collaboration 2-12. http://admdbsrv.ligo.caltech.edu/lsc_papers/default_reviewed.htf

[4] I. Chakrabarty, "Gravitational Waves: An Introduction", arXiv:physics/9908041 (1999).

[5] D. Sigg, "Gravitational Waves", Proceedings of TASI 98, Boulder, Colorado (1998) 1-11.

[6] M. Coles, "Introduction to Gravitational Waves", as slideshow for SURF 2001, (2001).

[7] "Gravitational radiation", Wikipedia (2005). http://en.wikipedia.org/wiki/Gravitational_waves

[8] E. E. Flanagan and S. A. Hughes, "The basics of gravitational wave theory", arXiv:gr-qc/0501041 (2005).