Gravitational Waves: An Introduction

Indrajit Chakrabarty*†

Abstract

In this article, I present an elementary introduction to the theory of gravitational waves. This article is meant for students who have had an exposure to general relativity, but, results from general relativity used in the main discussion has been derived and discussed in the appendices. The weak gravitational field approximation is first considered and the linearized Einstein's equations are obtained. We discuss the plane wave solutions to these equations and consider the transverse-traceless (TT) gauge. We then discuss the motion of test particles in the presence of a gravitational wave and their polarization. The method of Green's functions is applied to obtain the solutions to the linearized field equations in presence of a nonrelativistic, isolated source.

*Mehta Research Institute, Chhatnag Road, Jhusi. Allahabad. 211 019 INDIA
†E-mail: indrajit@mri.ernet.in

1 Introduction

In the past few years, research on the detection of gravitational waves has assumed new directions. Efforts are now underway to detect gravitational radiation from astrophysical sources, thereby enabling researchers to possess an additional tool to study the universe (See [6] for a recent review). According to Newton's theory of gravitation, the binary period of two point masses (e.g., two stars) moving in a bound orbit is strictly a constant quantity. However, Einstein's general theory of relativity predicts that two stars revolving around each other in a bound orbit suffer accelerations, and, as a result, gravitational radiation is generated. Gravitational waves carry away energy and momentum at the expense of the orbital decay of two stars, thereby causing the stars to gradually spiral towards each other and giving rise to shorter and shorter periods. This anticipated decrease of the orbital period of a binary pulsar was first observed in PSR 1913+16 by Taylor and Weisberg ([4]). The observation supported the idea of gravitational radiation first propounded in 1916 by Einstein in the Proceedings of the Royal Prussian Academy of Knowledge. Einstein showed that the first order contributon to the gravitational radiation must be quadrupolar in a particular coordinate system. Two years later, he extended his theory to all coordinate systems.

The weak nature of gravitational radiation makes it very difficult to design a sensitive detector. Filtering out the noisy background to isolate the useful signal requires great technical expertise. itself a field of research. Various gravitational wave detectors are fully/partially operational and we expect a certain result to appear from the observations in the near future.

This article gives an elementary introduction to the theory of gravitational waves. Important topics in general relativity including a brief introduction to tensors and a derivation of Einstein's field equations are discussed in the appendices. We first discuss the weak gravitational field approximation and obtain the linearized Einstein's field equations. We then discuss the plane wave solutions to these equations in vacuum and the restriction on them due to the transverse-traceless (TT) gauge. The motion of particles in the presence of gravitational waves and their polarization is then discussed in brief. We conclude by applying the method of Green's functions to show that gravitational radiation from matter at large distances is predominantly quadrupolar in nature.

2 The weak gravitational field approximation

Einstein's theory of general relativity leads to Newtonian gravity in the limit when the gravitational field is weak & static and the particles in the gravitational field move slowly. We now consider a less restrictive situation where the gravitational field is weak but not static, and there are no restrictions on the motion of particles in the gravitational field. In the absence of gravity, space-time is flat and is characterised by the Minkowski metric, $\eta_{\mu\nu}$. A weak gravitational field can be considered as a small 'perturbation' on the flat Minkowski metric[3],

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1 \tag{1}$$

Such coordinate systems are often called *Lorentz coordinate systems.* Indices of any tensor can be raised or lowered using $\eta^{\mu\nu}$ or $\eta_{\mu\nu}$ respectively as the corrections would be of higher order in the perturbation, $h_{\mu\nu}$. We can therefore write,

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \tag{2}$$

Under a background Lorentz transformation ([3]), the perturbation transforms as a second-rank tensor:

$$h_{\alpha\beta} = \Lambda_{\alpha}{}^{\mu}\Lambda_{\beta}{}^{\nu}\ h_{\mu\nu} \tag{3}$$

The equations obeyed by the perturbation, $h_{\mu\nu}$, are obtained by writing the Einstein's equations to first order. To the first order, the affine connection (See App. A) is,

$$\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2}\eta^{\lambda\rho}[\partial_{\mu}h_{\rho\nu} + \partial_{\nu}h_{\mu\rho} - \partial_{\rho}h_{\mu\nu}] + \mathcal{O}(h^2) \tag{4}$$

Therefore, the Riemann curvature tensor reduces to

$$R_{\mu\nu\rho\sigma} = \eta_{\mu\lambda}\partial_{\rho}\Gamma^{\lambda}{}_{\nu\sigma} - \eta_{\mu\lambda}\partial_{\sigma}\Gamma^{\lambda}{}_{\nu\rho} \tag{5}$$

The Ricci tensor is obtained to the first order as

$$R_{\mu\nu} \approx R^{(1)}_{\mu\nu} = \frac{1}{2}\left[\partial_{\lambda}\partial_{\nu}h^{\lambda}{}_{\mu} + \partial_{\lambda}\partial_{\mu}h^{\lambda}{}_{nu} - \partial_{\mu}\partial_{\nu}h - \Box h_{\mu\nu}\right] \tag{6}$$

where, $\Box \equiv \eta^{\lambda\rho}\partial_{\lambda}\partial_{\rho}$ is the D'Alembertian in flat space-time. Contracting again with $\eta^{\mu\nu}$, the Ricci scalar is obtained as

$$R = \partial_{\lambda}\partial_{\mu}h^{\lambda\mu} - \Box h \tag{7}$$

The Einstein tensor, $G_{\mu\nu}$, in the limit of weak gravitational field is

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R = \frac{1}{2}[\partial_\lambda\partial_\nu h^\lambda_\mu + \partial_\lambda\partial_\mu h^\lambda{}_\nu - \eta_{\mu\nu}\partial_\mu\partial_\nu h^{\mu\nu} + \eta_{\mu\nu}\Box h - \Box h_{\mu\nu}] \tag{8}$$

The linearised Einstein field equations are then

$$G_{\mu\nu} = 8\pi G T^{\mu\nu} \tag{9}$$

We can't expect the field equations (9) to have unique solutions as any solution to these equations will not remain invariant under a 'gauge' transformation. As a result, equations (9) will have infinitely many solutions. In other words, the decomposition (1) of $g_{\mu\nu}$ in the weak gravitational field approximation does not completely specify the coordinate system in space-time. When we have a system that is invariant under a gauge transformation, we *fix* the gauge and work in a selected coordinate system. One such coordinate system is the *harmonic coordinate system* ([5]). The gauge condition is

$$g^{\mu\nu}\Gamma^\lambda{}_{\mu\nu} = 0 \tag{10}$$

In the weak field limit, this condition reduces to

$$\partial_\lambda h^\lambda{}_\mu = \frac{1}{2}\partial_\mu h \tag{11}$$

This condition is often called the *Lorentz gauge*. In this selected gauge, the linearized Einstein equations simplify to,

$$\Box h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\Box h = -16\pi G T^{\mu\nu} \tag{12}$$

The 'trace-reversed' perturbation, $\bar{h}_{\mu\nu}$, is defined as ([3]),

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \tag{13}$$

The harmonic gauge condition further reduces to

$$\partial_\mu \bar{h}^\mu{}_\lambda = 0 \tag{14}$$

The Einstein equations are then

$$\Box \bar{h}_{\mu\nu} = -16\pi G T^{\mu\nu} \tag{15}$$

3 Plane-wave solutions and the transverse-traceless (TT) gauge

From the field equations in the weak-field limit, eqns.(15), we obtain the linearised field equations in vacuum,

$$\Box\bar{h}_{\mu\nu} = 0 \tag{16}$$

The vacuum equations for $\bar{h}_{\mu\nu}$ are similar to the wave equations in electromagnetism. These equations admit the plane-wave solutions,

$$\bar{h}_{\mu\nu} = A_{\mu\nu}\exp(\imath k_\alpha x^\alpha) \tag{17}$$

where, $A_{\mu\nu}$ is a constant, symmetric, rank-2 tensor and k_α is a constant four-vector known as the *wave vector*. Plugging in the solution (17) into the equation (16), we obtain the condition

$$k_\alpha k^\alpha = 0 \tag{18}$$

This implies that equation (17) gives a solution to the wave equation (16) if k_α is *null*; that is, tangent to the world line of a photon. This shows that gravitational waves propagate at the speed of light. The time-like component of the wave vector is often referred to as the *frequency* of the wave. The four-vector, k_μ is usually written as $k_\mu \equiv (\omega, \mathbf{k})$. Since k_α is null, it means that,

$$\omega^2 = |\mathbf{k}|^2 \tag{19}$$

This is often referred to as the *dispersion relation* for the gravitational wave. We can specify the plane wave with a number of independent parameters; 10 from the coefficients, $A_{\mu\nu}$ and three from the null vector, k_μ. Using the harmonic gauge condition (14), we obtain,

$$k_\alpha A^{\alpha\beta} = 0 \tag{20}$$

This imposes a restriction on $A^{\alpha\beta}$: it is orthogonal (*transverse*) to k_α. The number of independent components of $A_{\mu\nu}$ is thus reduced to six. We have to impose a gauge condition too as any coordinate transformation of the form

$$x^{\alpha'} = x^\alpha + \xi^\alpha(x^\beta) \tag{21}$$

will leave the harmonic coordinate condition

$$\Box x^\mu = 0 \tag{22}$$

satisfied as long as

$$\Box \xi^\alpha = 0 \tag{23}$$

Let us choose a solution

$$\xi_\alpha = C_\alpha \exp(\imath k_\beta x^\beta) \tag{24}$$

to the wave equation (23) for ξ_α. C_α are constant coefficients. We claim that this remaining freedom allows us to convert from the old constants, $A^{(\text{old})}_{\mu\nu}$, to a set of new constants, $A^{(\text{new})}_{\mu\nu}$, such that

$$A^{(\text{new})\ \ \mu}_{\mu} = 0 \quad (\textit{traceless}) \tag{25}$$

and

$$A_{\mu\nu} U^\beta = 0 \tag{26}$$

where, U^β is some fixed four-velocity, that is, any constant time-like unit vector we wish to choose. The equations (20), (25) and (26) together are called the *transverse traceless* (TT) gauge conditions ([3]). Thus, we have used up all the available gauge freedom and the remaining components of $A_{\mu\nu}$ must be physically important. The trace condition (25) implies that

$$\bar{h}^{TT}_{\alpha\beta} = h^{TT}_{\alpha\beta} \tag{27}$$

Let us now consider a background Lorentz transformation in which the vector U^α is the time basis vector $U^\alpha = \delta^\alpha_{\ 0}$. Then eqn.(26) implies that $A_{\mu 0} = 0$ for all μ. Let us orient the coordinate axes so that the wave is travelling along the z-direction, $k^\mu \rightarrow (\omega, 0, 0, \omega)$. Then with eqn.(26), eqn.(20) implies that $A_{\alpha z} = 0$ for all α. Thus, $A^{TT}_{\alpha\beta}$ in matrix form is

$$A^{TT}_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{xx} & A_{xy} & 0 \\ 0 & A_{xy} & -A_{xx} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{28}$$

4 Polarization of gravitational waves

In this section, we consider the effect of gravitational waves on free particles. Consider some particles described by a single velocity field, U^α and a separation vector, ζ^α. Then, the separation vector obeys the geodesic equation (See App. A)

$$\frac{d^2\zeta^\alpha}{d\tau^2} = R^\alpha_{\ \beta\gamma\delta} U^\beta U^\gamma \zeta^\delta \tag{29}$$

where, $U^\nu = dx^\nu/d\tau$ is the four-velocity of the two particles. We consider the lowest-order (flat-space) components of U^ν only since any corrections to U^ν that depend on $h_{\mu\nu}$ will give rise to terms second order in the perturbation in the above equation. Therefore, $U^\nu = (1, 0, 0, 0)$ and initially $\zeta^\nu = (0, \epsilon, 0, 0)$. Then to first order in $h_{\mu\nu}$, eqn. (29) reduces to

$$\frac{d^2\zeta^\alpha}{d\tau^2} = \frac{\partial^2\zeta^\alpha}{\partial t^2} = \epsilon R^\alpha{}_{00x} = -\epsilon R^\alpha{}_{0x0} \tag{30}$$

Using the definition of the Riemann tensor, we can show that in the TT gauge,

$$\begin{aligned} R^x{}_{0x0} = \quad R_{x0x0} = \quad & -\frac{1}{2} h^{TT}_{xx,00} \\ R^y{}_{0x0} = \quad R_{y0x0} = \quad & -\frac{1}{2} h^{TT}_{xy,00} \\ R^y{}_{0y0} = \quad R_{y0y0} = \quad & -\frac{1}{2} h^{TT}_{yy,00} = -R^x{}_{0x0} \end{aligned} \tag{31}$$

All other independent components vanish. This means that two particles initially separated in the x-direction have a separation vector which obeys the equation

$$\frac{\partial^2\zeta^x}{\partial t^2} = \frac{1}{2}\epsilon\frac{\partial^2}{\partial t^2}h^{TT}_{xx}, \frac{\partial^2\zeta^y}{\partial t^2} = \frac{1}{2}\epsilon\frac{\partial^2}{\partial t^2}h^{TT}_{xy} \tag{32}$$

Similarly, two particles initially separated by ϵ in the y-direction obey the equations

$$\begin{aligned} \frac{\partial^2\zeta^y}{\partial t^2} = \frac{1}{2}\epsilon\frac{\partial^2}{\partial t^2}h^{TT}_{yy} &= -\frac{1}{2}\epsilon\frac{\partial^2}{\partial t^2}h^{TT}_{xx} \\ \frac{\partial^2\zeta^x}{\partial t^2} &= \frac{1}{2}\epsilon\frac{\partial^2}{\partial t^2}h^{TT}_{xy} \end{aligned} \tag{33}$$

We can now use these equations to describe the polarization of a gravitational wave. Let us consider a ring of particles initially at rest as in Fig. 1(a). Suppose a wave with $h^{TT}_{xx} \neq 0, h^{TT}_{xy} = 0$ hits them. The particles respond to the wave as shown in Fig. 1(b). First the particles along the x-direction come towards each other and then move away from each other as h^{TT}_{xx} reverses sign. This is often called + polarization. If the wave had $h^{TT}_{xy} \neq 0$, but, $h^{TT}_{xx} = h^{TT}_{yy} = 0$, then the particles respond as shown in Fig. 1(c). This is known as $\times$ polarization. Since h^{TT}_{xy} and h^{TT}_{xx} are independent, the figures 1(b) and 1(c) demonstrate the existence of two different states of polarisation. The two states of polarisation are oriented at an angle of 45^o to each other unlike in electromagnetic waves were the two states of polarization.

5 Generation of gravitational waves

In section III, we obtained the plane wave solutions to the linearized Einstein's equations in vacuum, eqns.(16). To obtain the solution of the linearised equations (15), we will use the Green's function method. The Green's function, $G(x^\mu - y^\mu)$, of the D'Alembertian operator $\Box$, is the solution of the wave equation in the presence of a delta function source:

$$\Box\, G(x^\mu - y^\mu) = \delta^{(4)}(x^\mu - y^\nu) \tag{34}$$

where $\delta^{(4)}$ is the four-dimensional Dirac delta function. The general solution to the linearized Einstein's equations (15) can be written using the Green's function as

$$\bar{h}_{\mu\nu}(x^\alpha) = -16\pi G \int d^4y\ G(x^\alpha - y^\alpha) T_{\mu\nu}(y^\alpha) \tag{35}$$

The solutions to the eqn.(34) are called *advanced* or *retarded* according as they represent waves travelling backward or forward in time, respectively. We are interested in the retarded Green's function as it represents the net effect of signals from the past of the point under consideration. It is given by

$$G(x^\mu - y^\mu) = -\frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\delta\left[|\mathbf{x}-\mathbf{y}| - (x^0 - y^0)\right] \times\ \theta(x^0 - y^0) \tag{36}$$

where, $\mathbf{x} = (x^1, x^2, x^3)$ and $\mathbf{y} = (y^1, y^2, y^3)$ and $|\mathbf{x}-\mathbf{y}| = [\delta_{ij}(x^i - y^i)(x^j - y^j)]^{1/2}$. $\theta(x^0 - y^0)$' is the Heaviside unit step function, it equals 1 when $x^0 > y^0$, and equals 0 otherwise. Using the relation (36) in (35), we can perform the integral over y^0 with the help of the delta function,

$$\bar{h}_{\mu\nu}(t, \mathbf{x}) = 4G \int d^3y\ \frac{1}{|\mathbf{x}-\mathbf{y}|} T_{\mu\nu}(t - |\mathbf{x}-\mathbf{y}|, \mathbf{y}) \tag{37}$$

where $t = x^0$. The quantity

$$t_{\mathrm{R}} = t - |\mathbf{x}-\mathbf{y}| \tag{38}$$

is called the *retarded time*. From the expression (37) for $\bar{h}_{\mu\nu}$, we observe that the disturbance in the gravitational field at $(t, \mathbf{x})$ is a sum of the influences from the energy and momentum sources at the point $(t_{\mathrm{R}}, \mathbf{y})$ on the past light cone.

Using the expression (37), we now consider the gravitational radiation emitted by an isolated far away source consisting of very slowly moving particles (the spatial extent of the source is negligible compared to the distance between the source and the observer). The Fourier transform of the perturbation $\bar{h}_{\mu\nu}$ is

$$\tilde{\bar{h}}_{\mu\nu}(\omega, \mathbf{x}) = \frac{1}{\sqrt{2\pi}} \int dt\, \exp(\iota\omega t)\; \bar{h}_{\mu\nu}(t, \mathbf{x}) \tag{39}$$

Using the expression (37) for $\bar{h}_{\mu\nu}(t, \mathbf{x})$, we get

$$\tilde{\bar{h}}_{\mu\nu} = 4G \int d^3y\, \exp(\iota\omega|\mathbf{x} - \mathbf{y}|)\; \frac{\tilde{T}^{\mu\nu}(\omega, \mathbf{y})}{|\mathbf{x} - \mathbf{y}|} \tag{40}$$

Under the assumption that the spatial extent of the source is negligible compared to the distance between the source and the observer, we can replace the term $\exp(\iota\omega|\mathbf{x} - \mathbf{y}|)/|\mathbf{x} - \mathbf{y}|$ in (40) by $\exp(\iota\omega \mathrm{R})/\mathrm{R}$. Therefore,

$$\tilde{\bar{h}}_{\mu\nu}(\omega, \mathbf{x}) = 4G\; \frac{\exp(\iota\omega \mathrm{R})}{\mathrm{R}} \; \int d^3y\; \tilde{T}_{\mu\nu}(\omega, \mathbf{y}) \tag{41}$$

The harmonic gauge condition (14) in Fourier space is

$$\partial_\mu \bar{h}^{\;\mu\nu}(t, \mathbf{x}) = \partial_\mu \int d\omega\; \tilde{\bar{h}}^{\;\mu\nu} \;\exp(-\iota\omega t) = 0 \tag{42}$$

Separating out the space and time components,

$$\partial_0 \int d\omega\; \tilde{\bar{h}}^{0\nu}(\omega, \mathbf{x})\exp(-\iota\omega t) + \partial_i \int d\omega\; \tilde{\bar{h}}^{i\nu}(\omega, \mathbf{x})\exp(-\iota\omega t) = 0 \tag{43}$$

Or,

$$\iota\omega\tilde{\bar{h}}^{0\nu} \;=\; \partial_i\tilde{\bar{h}}^{i\nu} \tag{44}$$

Thus, in eqn.(41), we need to consider the space-like components of $\tilde{\bar{h}}_{\mu\nu}(\omega, \mathbf{y})$. Consider,

$$\int d^3y\; \partial_k \left(y^i \tilde{T}_{kj}\right) = \int d^3y\;\; \left(\partial_k y^i\right) \tilde{T}^{kj} + \int d^3y\; y^i \left(\partial_k \tilde{T}^{kj}\right)$$

On using Gauss' theorem, we obtain,

$$\int d^3y\ \tilde{T}^{ij}(\omega,\mathbf{y}) = -\int d^3y\ y^i \left(\partial_k \tilde{T}^{kj}\right) \tag{45}$$

Consider the Fourier space version of the conservation equation for $T^{\mu\nu}$, viz., $\partial_\mu T^{\mu\nu}(t,\mathbf{x}) = 0$. Separating the time and space components of the Fourier transform equation, we have,

$$\partial_i \tilde{T}^{i\nu} = \iota\omega T^{0\nu} \tag{46}$$

Therefore,

$$\int d^3y\ \tilde{T}^{ij}(\omega,\mathbf{y}) = \iota\omega \int d^3y\ y^i\ \tilde{T}^{0j} = \frac{\iota\omega}{2}\int\ d^3y\ \left(y^i\ \tilde{T}^{0j} + y^j\ \tilde{T}^{0i}\right) \tag{47}$$

Consider

$$\int\ d^3y\, \partial_l \left(y_i\ y_j \tilde{T}^{0l}\right) = \int\ d^3y \left[\left(\partial_l y^i\right) y^j + \left(\partial_l y^j\right) y^i\right] \tilde{T}^{0l} + \int d^3y\, y^i\, y^j \left(\partial_l \tilde{T}^{0l}\right) \tag{48}$$

Simplifying the equation above, we obtain for the left hand side

$$\int d^3y\ \left(y^i\ \tilde{T}^{0j} + y^j \tilde{T}^{0i}\right) + \int d^3y\ y^i\ y^j \left(\partial_l \tilde{T}^{0l}\right)$$

Since the term on the left hand side of eqn.(47) is a surface term, it vanishes and we obtain

$$\int d^3y \left(y^i \tilde{T}^{0j} + y^j \tilde{T}^{0i}\right) = -\int d^3y\ y^i\ y^j \left(\partial_l \tilde{T}^{0l}\right) \tag{49}$$

Using the equations (46) and (48), we can write,

$$\int d^3y\ \tilde{T}^{ij}(\omega,\mathbf{y}) = \frac{\iota\omega}{2}\int d^3y\ \partial_l \left(y^i y^j \tilde{T}^{0l}\right) \tag{50}$$

Using the eqn(45), we can write,

$$\int d^3y\ \tilde{T}^{ij}(\omega,\mathbf{y}) = -\frac{\omega^2}{2}\int d^3y\ y^i y^j\ \tilde{T}^{00} \tag{51}$$

We define the *quadrupole moment tensor* of the energy density of the source as

$$\tilde{q}_{ij}(\omega) = 3\int d^3y\ y^i y^j\ \tilde{T}^{00}(\omega,\mathbf{y}) \tag{52}$$

In terms of the quadrupole moment tensor, we have

$$\int d^3y\ \tilde{T}^{ij}(\omega, \mathbf{y}) = -\frac{\omega^2}{6}\ \tilde{q}_{ij}(\omega) \tag{53}$$

Therefore, the solution (41) becomes

$$\tilde{\bar{h}}_{ij}(\omega, \mathbf{x}) = 4G\ \frac{\exp(\iota\omega \mathrm{R})}{\mathrm{R}} \left(-\ \frac{\omega^2}{6} \tilde{q}_{ij}(\omega) \right) \tag{54}$$

Simplifying further,

$$\tilde{\bar{h}}_{ij}(\omega, \mathbf{x}) = -\frac{2}{3}\ \frac{G\omega^2}{\mathrm{R}}\ \exp(\iota\omega \mathrm{R})\ \tilde{q}_{ij}(\omega) \tag{55}$$

Taking the Fourier transform of eqn.(54), and simplifying, we finally obtain for the perturbation

$$\bar{h}_{ij}(t, \mathbf{x}) = \frac{2G}{3\mathrm{R}}\ \frac{d}{dt^2}\ q_{ij}(t_{\mathrm{R}}) \tag{56}$$

where, $t_{\mathrm{R}} = t - |\mathbf{x} - \mathbf{y}|$ is the retarded time. In the expression (54), we see that the gravitational wave produced by an isolated, monochromatic and non-relativistic source is therefore proportional to the second derivative of the quadrupole moment of the energy density at the point where the past light cone of the observer intersects the cone. The quadrupolar nature of the wave shows itself by the production of shear in the particle distribution, and there is zero average translational motion. The leading contribution to electromagnetic radiation comes from the changing dipole moment of the charge density. This remarkable difference in the nature of gravitational and electromagnetic radiation arises from the fact that the centre of mass of an isolated system can't oscillate freely but the centre of charge of a charge distribution can. The quadrupole momnet of a system is generally smaller than the dipole moment and hence gravitational waves are weaker than electromagnetic waves.

6 Epilogue

This lecture note on gravitational waves leaves several topics untouched. There are a number of good references on gravitation where the inquisitive reader can find more about gravitational waves and their detection. I have freely drawn from various sources and I don't claim any originality in this work. I hope I have been able to incite some interest in the reader about a topic on which there is a dearth of literature.

Acknowledgements

This expository article grew out of a seminar presented at the end of the *Gravitation and Cosmology* course given by Prof. Ashoke Sen. I am grateful to all my colleagues who helped me during the preparation of the lecture.

Appendix A: Some topics in general theory of relativity

An event in relativity is characterised by a set of coordinates (t, x, y, z) in a definite coordinate system. Transformations between the coordinates of an event observed in two different reference frames are called *Lorentz transformations*. These transformations mix up space and time and hence the coordinates are redefined so that all of them have dimensions of length. We write $x^0 \equiv ct, x^1 \equiv x, x^2 \equiv y, x^3 \equiv z$ and a general component of a four vector (x^0, x^1, x^2, x^3) as x^μ. A Lorentz transformation is then written as

$$x^\mu = \Lambda^\mu{}_\nu x^\nu \tag{57}$$

where,

$$\Lambda = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{58}$$

At this point, it is useful to note the *Einstein summation convention*: whenever an index appears as a subscript and as a superscript in an expression, we sum over all values taken by the index. Under a Lorentz transformation, the spacetime interval $-(ct)^2 + x^2 + y^2 + z^2$ remains invariant. The length of a four-vector is given by

$$|\mathbf{x}| = -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 \tag{59}$$

We never extract a square root of the expression (59) since $|\mathbf{x}|$ can be negative. Four-vectors that have negative length are called *time-like*, while those with positive lengths are called *space-like*. Four-vectors with zero length are called *null*. The notion of "norm" of a four-vector is introduced with the help of the *Minkowski metric*:

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{60}$$

Then, we have,

$$|\mathbf{x}| = x^\mu \eta_{\mu\nu} x^\nu \tag{61}$$

There are two kinds of vectors that are classified in the way they transform under a Lorentz transformation:

$$\begin{aligned} \text{Contravariant} \quad &:x^{\mu} &= \Lambda_{\nu}{}^{\mu} x^{\nu} \\ \text{Covariant} \quad &:x_{\mu} &= \Lambda_{\mu}{}^{\nu} x_{\nu} \end{aligned} \tag{62}$$

Vectors are *tensors* of rank one. $\eta^{\mu\nu}(\eta_{\mu\nu})$ is called the metric tensor; it is a tensor of rank two. There are other higher rank tensors which we will encounter later. If two coordinate systems are linked by a Lorentz transformation as:

$$x'^{\ \nu} = \Lambda^{\nu}{}_{\mu} x^{\mu} \tag{63}$$

then, multiplying both sides of the equation above by $\Lambda_{\nu}{}^{\kappa}$ and differentiating, we get,

$$\frac{\partial x^{\kappa}}{\partial x'^{\ \nu}} = \Lambda_{\nu}{}^{\kappa} \tag{64}$$

Therefore, we see that

$$\frac{\partial}{\partial x'^{\ \mu}} = \Lambda_{\mu}{}^{\nu} \frac{\partial}{\partial x^{\nu}} \tag{65}$$

Thus,

$$\partial_{\mu} \equiv \partial/\partial x^{\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \tag{66}$$

transforms as a covariant vector. The differential operates on tensors to yield higher-rank tensors. A scalar s can be constructed using the Minkowski metric and two four-vectors u^{μ} and v^{ν} as:

$$s = \eta_{\mu\nu} u^{\mu} v^{\nu} \tag{67}$$

A scalar is an invariant quantity under Lorentz transformations. Using the chain rule,

$$dx'^{\ \mu} = \frac{\partial x'^{\ \mu}}{\partial x^{\nu}} dx^{\nu} \tag{68}$$

we have,

$$s = \left(\eta_{\mu\nu} \frac{\partial x^{\mu}}{\partial x'^{\kappa}} \frac{\partial x^{\nu}}{\partial x'^{\lambda}} \right) u'^{\kappa} v'^{\lambda} \tag{69}$$

If we define

$$g_{\kappa\lambda} \equiv \eta_{\mu\nu} \frac{\partial x^{\mu}}{\partial x'^{\kappa}} \frac{\partial x^{\nu}}{\partial x'^{\lambda}} \tag{70}$$

then,

$$s = g_{\kappa\lambda} \ \ u'^{\kappa} v'^{\lambda} \tag{71}$$

$g_{\kappa\lambda}$ is called the metric tensor; it is a symmetric, second-rank tensor.

To follow the motion of a freely falling particle, an *inertial* coordinate system is sufficient. In an inertial frame, a particle at rest will remain so if no forces act on it. There is a frame where particles move with a uniform velocity. This is the frame which falls freely in a gravitational field. Since this frame accelerates at the same rate as the free particles do, it follows that all such particles will maintain a uniform velocity relative to this frame. Uniform gravitational fields are equivalent to frames that accelerate uniformly relative to inertial frames. This is the *Principle of Equivalence* between gravity and acceleration. The principle just stated is known as the *Weak Equivalence Principle* because it only refers to gravity.

In treating the motion of a particle in the presence of gravity, we define the *Christoffel symbol* or the *affine connection* as

$$\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left(\frac{\partial g_{\nu\alpha}}{\partial x^{\beta}} + \frac{\partial g_{\beta\nu}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\nu}} \right) \tag{72}$$

Γ plays the same role for the gravitational field as the field strength tensor does for the electromagnetic field. Using the definition of affine connection, we can obtain the expression for the covariant derivative of a tensor:

$$\mathcal{D}_{\kappa} A^{\nu} \equiv \frac{\partial A^{\nu}}{\partial x^{\kappa}} + \Gamma^{\nu}{}_{\kappa\nu} A^{\alpha} \tag{73}$$

It is straightforward to conclude that the covariant derivative of the metric tensor vanishes. The concept of "parallel transport" of a vector has important implications. We can't define globally parallel vector fields. We can define local parallelism. In the Euclidean space, a straight line is the only curve that parallel transports its own tangent vector. In curved space, we can draw "nearly" straight lines by demanding parallel transport of the tangent vector. These "lines" are called *geodesics*. A geodesic is a curve of extremal length between any two points. The equation of a geodesic is

$$\frac{d^2 x^{\alpha}}{d\lambda^2} + \Gamma^{\alpha}{}_{\mu\beta} \frac{dx^{\mu}}{d\lambda} \frac{dx^{\beta}}{d\lambda} = 0 \tag{74}$$

The parameter λ is called an *affine* parameter. A curve having the same path as a geodesic but parametrised by a non-affine parameter is not a geodesic curve. The Riemannian curvature tensor is defined as

$$R^{\mu}{}_{\gamma\alpha\nu} = \frac{\partial \Gamma^{\mu}{}_{\alpha\gamma}}{\partial x^{\nu}} - \frac{\partial \Gamma^{\mu}{}_{\nu\gamma}}{\partial x^{\alpha}} + \Gamma^{\mu}{}_{\nu\beta}\, \Gamma^{\beta}{}_{\alpha\gamma} - \Gamma^{\mu}{}_{\alpha\beta}\, \Gamma^{\beta}{}_{\nu\gamma} \tag{75}$$

In a “flat” space,

$$R^{\mu}{}_{\gamma\alpha\nu} = 0 \tag{76}$$

Geodesics in a flat space maintain their separation; those in curved spaces don’t. The equation obeyed by the separation vector ζ^α in a vector field V is

$$\mathcal{D}_V\ \mathcal{D}_V\ \zeta^\alpha = R^{\mu}{}_{\gamma\alpha\nu} V^\mu\ V^\nu\ \zeta^\beta \tag{77}$$

If we differentiate the Riemannian curvature tensor and permute the indices, we obtain the *Bianchi identity*:

$$\partial_\lambda R_{\alpha\beta\mu\nu} + \partial_\nu R_{\alpha\beta\lambda\mu} + \partial_\mu R_{\alpha\beta\nu\lambda} = 0 \tag{78}$$

Since in an inertial coordinate system the affine connection vanishes, the equation above is equivalent to one with partials replaced by covariant derivatives. The *Ricci tensor* is defined as

$$R_{\alpha\beta} \equiv R^{\mu}{}_{\alpha\mu\beta} = R_{\beta\alpha} \tag{79}$$

It is a symmetric second rank tensor. The *Ricci scalar* (also known as *scalar curvature*) is obtained by a further contraction,

$$R \equiv R^{\beta}{}_{\beta} \tag{80}$$

The *stress-energy tensor* (also called *energy-momentum tensor*) is defined as the flux of the α-momentum across a surface of constant x^β. In component form, we have:

1. T^{00} = Energy density = ρ
2. T^{0i} = Energy flux (Energy may be transmitted by heat cinduction)
3. T^{i0} = Momentum density (Even if the particles don’t carry momentum, if heat is being conducted, then the energy will carry momentum)
4. T^{ij} = Momentum flux (Also called *stress*)

Appendix B: The Einstein field equation

The curvature of space-time is necessary and sufficient to describe gravity. The latter can be shown by considering the Newtonian limit of the geodesic equation. We require that

- the particles are moving slowly with respect to the speed of light,
- the gravitational field is weak so that it may be considered as a perturbation of flat space, and,
- the gravitational field is static.

In this limit, the geodesic equation changes to,

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}(\frac{dt}{d\tau})^2 = 0 \tag{81}$$

The affine connection also simplifies to

$$\Gamma^\mu_{00} = -\frac{1}{2}g^{\mu\lambda}\partial_\lambda g_{00} \tag{82}$$

In the weak gravitational field limit, we can lower or raise the indices of a tensor using the Minkowskian flat metric, e.g.,

$$\eta^{\mu\nu}h_{\mu\rho} = h^\nu{}_\rho \tag{83}$$

Then, the affine connection is written as

$$\Gamma^\mu_{00} = -\frac{1}{2}\eta^{\mu\lambda}\partial_\lambda h_{00} \tag{84}$$

The geodesic equation then reduces to

$$\frac{d^2x^\mu}{d\tau^2} = \frac{1}{2}\eta^{\mu\lambda}\left(\frac{dt}{d\tau}\right)^2\partial_\lambda h_{00} \tag{85}$$

The space components of the above equation are

$$\frac{d^2x^i}{d\tau^2} = \frac{1}{2}(\frac{dt}{d\tau})^2\partial_i h_{00} \tag{86}$$

Or,

$$\frac{d^2x^i}{dt^2} = \frac{1}{2}\partial_i h_{00} \tag{87}$$

The concept of an *inertial mass* arises from Newton's second law:

$$\mathbf{f} = m_i \mathbf{a} \tag{88}$$

According to the the law of gravitation, the gravitational force exerted on an abject is proportional to the gradient of a scalar field Φ, known as the scalar gravitational potential. The constant of proportionality is the gravitational mass, m_g:

$$\mathbf{f}_g = -m_g \nabla\Phi \tag{89}$$

According to the Weak Equivalence Principle, the inertial and gravitational masses are the same,

$$m_i = m_g \tag{90}$$

And, hence,

$$\mathbf{a} = -\nabla\Phi \tag{91}$$

Comparing equations (86) and (91), we find that they are the same if we identify,

$$h_{00} = -2\Phi \tag{92}$$

Thus,

$$g_{00} = -(1 + 2\Phi) \tag{93}$$

The curvature of space-time is sufficient to describe gravity in the Newtonian limit as along as the metric takes the form (93). All the basic laws of Physics, beyond those governing freely-falling particles adapt to the curvature of space-time (that is, to the presence of gravity) when we are working in Riemannian normal coordinates. The tensorial form of any law is coordinate-independent and hence, translating a law into the language of tensors (that is, to replace the partial derivatives by the covariant derivatives), we will have an universal law which holds in all coordinate systems. This procedure is sometimes called the Principle of Equivalence. For example, the conservation equation for the energy-momentum tensor $T^{\mu\nu}$ in flat space-time, viz.,

$$\partial_\mu T^{\mu\nu} = 0 \tag{94}$$

is immediately adapted to the curved space-time as,

$$D_\mu T^{\mu\nu} = 0 \tag{95}$$

This equation expresses the conservation of energy in the presence of a gravitational field. We can now introduce Einstein's field equations which governs how the metric responds to energy and momentum. We would like to derive an equation which will supercede the Poisson equation for the Newtonian potential:

$$\nabla^2\Phi = -4\pi G\rho \tag{96}$$

where, $\nabla^2 = \delta^{ij}\partial_i\partial_j$ is the Laplacian in space and ρ is the mass density. A relativistic generalisation of this equation must have a tensorial form so that the law is valid in all coordinate systems. The tensorial counterpart of the mass density is the energy-momentum tensor, $T^{\mu\nu}$. The gravitational potential should get replaced by the metric. Thus, we guess that our new equation will have $T^{\mu\nu}$ set proportional to some tensor which is second-order in the derivatives of the metric,

$$T^{\mu\nu} = \kappa A_{\mu\nu} \tag{97}$$

where, $A_{\mu\nu}$ is the tensor to be found. The requirements on the equation above are:-

- By definition, the R.H.S must be a second-rank tensor.
- It must contain terms linear in the second derivatives or quadratic in the first derivative of the metric.
- The R.H.S must be symmetric in μ and ν as $T^{\mu\nu}$ is symmetric.
- Since $T^{\mu\nu}$ is conserved, the R.H.S must also be conserved.

The first two conditions require the right hand side to be of the form

$$\alpha R_{\mu\nu} + \beta R g_{\mu\nu} = T_{\mu\nu} \tag{98}$$

where $R_{\mu\nu}$ is the Ricci tensor, R is the scalar curvature and α & β are constants. This choice is symmetric in μ and ν and hence satisfies the third condition. From the last condition, we obtain

$$g^{\nu\sigma}D_\sigma(\alpha R_{\mu\nu} + \beta R g_{\mu\nu}) = 0 \tag{99}$$

This equation can't be satisfied for arbitrary values of α and β. This equation holds only if α/β is fixed. As a consequence of the Bianchi identity, viz.,

$$D^\mu R_{\mu\nu} = \frac{1}{2}D_\nu R \tag{100}$$

we choose,

$$\beta = -\frac{1}{2}\alpha \tag{101}$$

With this choice, the equation (42) becomes

$$\alpha(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}) = T^{\mu\nu} \tag{102}$$

In the weak field limit,

$$g_{00} \approx -2\Phi \tag{103}$$

the 00-component of the equation(42), viz.,

$$-\alpha\nabla^2 g_{00} = T_{00} \Rightarrow 2\alpha\nabla^2\Phi = \rho \tag{104}$$

Compare this result with Newtons equation (40), we obtain,

$$2a = \frac{1}{4\pi G} \tag{105}$$

Thus, we obtain the Einstein field equations in their final form as

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T^{\mu\nu} \tag{106}$$

References

[1] S. Caroll, *Lecture notes on General Relativity*, gr-qc/9712019

[2] L. D. Landau & E. M. Lifshitz, *Classical Theory of Fields*, Pergamon Press (1980)

[3] B. F. Schutz, *A first course in general relativity*, Cambridge University Press (1995)

[4] J. H. Taylor & J. M. Weisberg, *ApJ*, **253**, 908 (1982)

[5] S. Weinberg, *Gravitation and Cosmology*, Princeton, N. Y.

[6] R. Weiss, *Rev. Mod. Phys.*, **71**, S187 (1999)

Figures

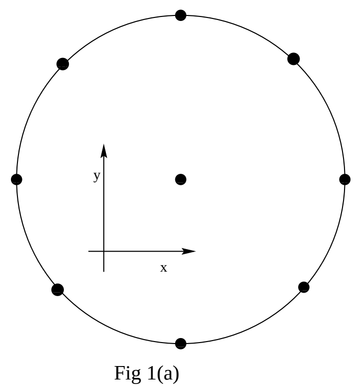

Figure 1: The initial configuration of test particles on a circle before a gravitational wave hits them.

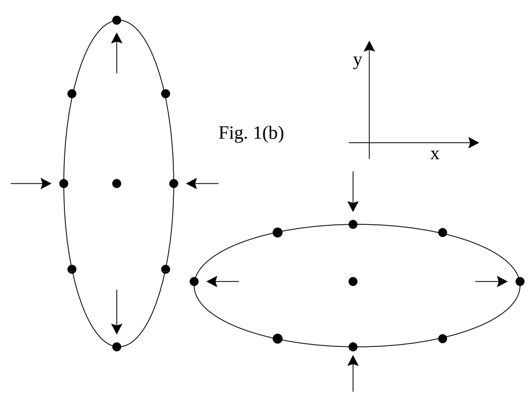

Figure 2: Displacement of test particles caused by the passage of a gravitational wave with the + polarization. The two states are separated by a phase difference of π.

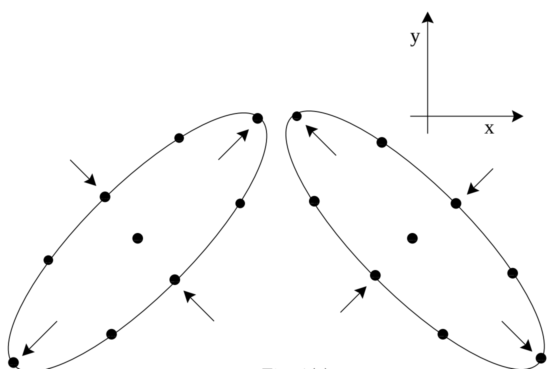

Figure 3: Displacement of test particles caused by the passage of a gravitational wave with the $\times$ polarization.